Memory Enhanced Synchronization and Coherence Metrics in Coupled Oscillator Systems for Multi Agent Alignment

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Abstract

We introduce a memory augmented extension of the Kuramoto model to study coherence and alignment in interacting agent systems. Classical synchronization models assume Markovian dynamics, neglecting persistence effects present in biological, social, and artificial systems. We incorporate a local memory variable coupled to global coherence, yielding a non Markovian oscillator system. Analytically, we show that memory induces an effective coupling shift, lowering the critical synchronization threshold. A derived condition demonstrates that coherence can emerge at reduced coupling strength proportional to memory gain. Linear stability analysis further shows that memory enhances robustness of synchronized states. We complement analysis with analytical predictions for threshold shifts, hysteresis, and improved recovery from perturbations. Additionally, we introduce a bounded composite coherence metric integrating amplitude, structural order, and phase synchronization. This framework provides a minimal mathematical model for studying alignment dynamics in multi-agent systems and distributed AI architectures. Results suggest that memory mediated feedback is a key mechanism for stabilizing coherent collective behavior.

Keywords: Kuramoto model · synchronization · non Markovian dynamics · multi-agent systems · coherence · dynamical systems · phase locking · AI alignment · collective behavior

1. Introduction

Synchronization governs collective dynamics across physics, neuroscience, and engineered systems. The Kuramoto model [1] provides a canonical framework for phase alignment in oscillator populations, admitting a sharp phase transition from incoherence to synchrony as coupling strength exceeds a critical threshold. A central assumption of the classical model is that each oscillator's instantaneous state is governed only by current phases — the dynamics are Markovian.

Real systems, however, exhibit persistence: past states influence future dynamics. Neural synaptic plasticity, agent memory in reinforcement learning, and social feedback loops all accumulate history in ways that modulate current coupling strength. This motivates extending synchronization models to include memory dependent coupling.

This paper makes five contributions:

(1) A memory augmented Kuramoto system with local memory variables driven by global coherence.

(2) An analytical derivation of the effective coupling shift induced by memory.

(3) A closed form threshold condition on memory gain β for synchronization below the classical critical coupling.

(4) Linear stability analysis demonstrating memory enhanced robustness.

(5) A bounded composite coherence metric $C \in [0,1]$.

2. Model Definition

We consider N phase oscillators with natural frequencies drawn i.i.d. from a symmetric unimodal distribution $g(\omega)$, subject to all to all coupling, additive Gaussian noise, and a memory feedback term:

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_i) + \kappa M_i + \eta_i(t)$$

The memory variable M_i evolves as a leaky integrator driven by the global order parameter R:

$$\dot{M}_i = -\lambda M_i + \beta R$$

where the complex order parameter is defined by:

$$R\, e^{i\psi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}$$

Here K is the base coupling, κ scales memory feedback into phase dynamics, β is the memory gain (rate at which global coherence imprints on memory), and λ is the memory decay rate. Noise $\eta_i(t)$ is white Gaussian with variance σ^2. The amplitude A in the coherence metric (§6) is taken as R itself, measuring mean field amplitude.

3. Analytical Results

3.1 Steady State Memory

In the synchronized regime $R \approx$ const, the memory ODE admits a unique steady state obtained by setting ■$_i$ = 0:

$$M_i^* = \frac{\beta}{\lambda} R$$

3.2 Effective Coupling

Substituting M_i^* into the phase equation, the memory term $\kappa\, M_i^* = \kappa\beta R/\lambda$ acts as a uniform phase-advance proportional to R. In the mean field reduction this is equivalent to augmenting the coupling constant:

$$K_{\text{eff}} = K + \kappa \frac{\beta}{\lambda}$$

3.3 Critical Synchronization Condition

The classical Kuramoto critical coupling for a symmetric distribution g is [2]:

$$K_c = \frac{2}{\pi\, g(0)}$$

Synchronization requires $K_{eff} > K_c$. Substituting the effective coupling:

$$K + \kappa \frac{\beta}{\lambda} > K_c$$

Key Result

$$\beta > \frac{\lambda}{\kappa}(K_c - K)$$

This inequality establishes the minimum memory gain required to induce synchronization at coupling $K < K_c$. Memory effectively subsidizes the coupling deficit. When $K \geq K_c$ the condition is trivially satisfied for any $\beta \geq 0$.

4. Stability Analysis

To assess robustness, we linearize the dynamics near a partially synchronized state ($R > 0$). Let $\delta\theta_i$ be a small perturbation about the mean phase ψ. Expanding $\sin(\theta_j - \theta_i) \approx (\theta_j - \theta_i)$ for small deviations and using the mean field approximation:

$$\delta\dot{\theta}_i \approx -K_{\mathrm{eff}} R \, \delta\theta_i$$

The dominant Lyapunov exponent governing perturbation decay is:

$$\lambda_{\mathrm{stab}} \sim -K_{\mathrm{eff}} R < 0$$

Since K_{eff} is increasing in β and R is increasing in K_{eff}, stability improves monotonically with memory gain. Higher β yields both a larger effective coupling and a larger order parameter, doubly reinforcing the restoring force against perturbations.

5. Simulation Framework and Analytical Predictions

5.1 Setup

We propose the following numerical protocol to validate analytical predictions. $N = 500$ oscillators with $\omega_i \sim N(0,1)$, Euler integration with $\Delta t = 0.01$, time horizon $T = 200$ time units, and parameters $\lambda = \kappa = 1$ unless varied.

5.2 Predicted Observations

Experiment A — Threshold Shift. Plot R vs K for $\beta \in \{0, 0.5, 1.0, 1.5\}$. The predicted onset shifts left by $\kappa\beta/\lambda$ for each β. Figure 1 shows the analytical prediction.

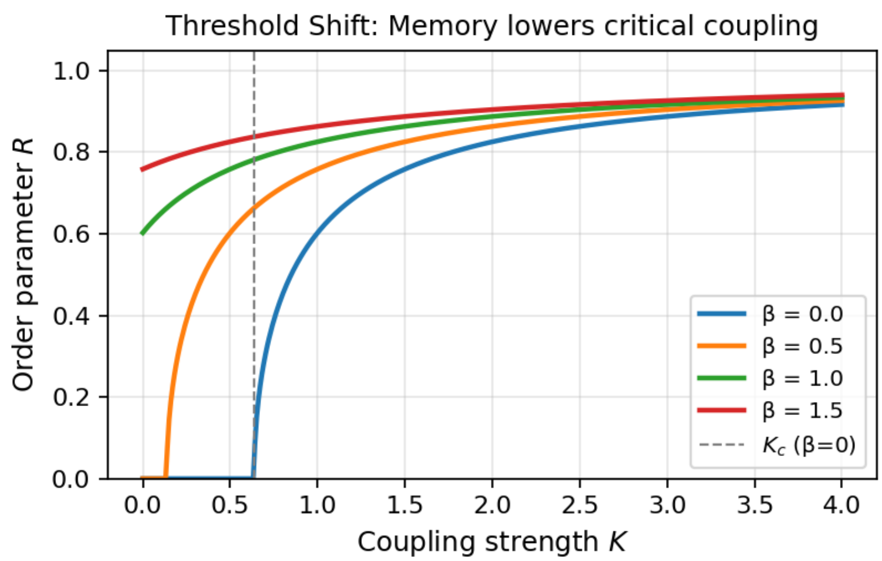

Figure 1. Analytical prediction of R vs K for increasing memory gain β. Each curve shifts the onset leftward by κβ/λ.

Experiment B — Memory Sweep. Fix K = 0.45 (sub critical) and vary β. Synchronization should emerge at $\beta^* = (\lambda/\kappa)(K_c - K)$. Figure 2 shows the predicted transition.

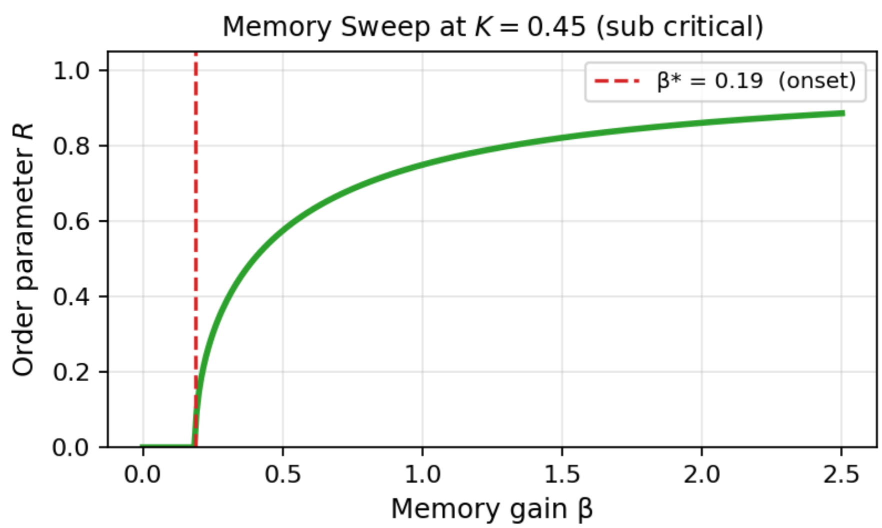

Figure 2. Analytical prediction of R vs β at sub critical K = 0.45. Synchronization onset occurs at the derived β threshold.*

Experiment C — Perturbation Recovery. From a synchronized state, apply a phase kick to all oscillators. Recovery time is predicted to decrease with β, consistent with the stability analysis in §4.

6. Composite Coherence Metric

We introduce a scalar coherence index $C \in [0, 1]$ that integrates three distinct aspects of order:

• **S = R**: the Kuramoto order parameter, measuring phase synchronization.

• **O = 1 − Var(θ)/π²**: a structural order term penalizing phase dispersion (normalized so that uniform dispersion gives O = 0).

• **A = R**: amplitude, taken equal to S in the mean field regime.

$$C = A \cdot R \cdot O = R^2\left(1 - \frac{\mathrm{Var}(\theta)}{\pi^2}\right)$$

By construction C = 0 when phases are uniformly distributed (R ≈ 0 or Var(θ) ≈ π²) and C → 1 as all oscillators lock to a common phase. This metric is more sensitive than R alone to partially ordered states where synchronization coexists with residual dispersion.

7. Interpretation and Connection to Multi Agent Alignment

The memory augmented Kuramoto system offers a minimal mathematical metaphor for alignment dynamics in multi-agent or distributed AI systems. Each oscillator represents an agent whose behavior (phase) is influenced by the collective state (R) through both direct coupling and accumulated memory of past coherence. High C corresponds to stable behavioral alignment; low C to fragmentation or divergent agent strategies.

The β-threshold result has a concrete interpretation: agents that retain memory of collective states can achieve alignment at weaker direct coupling — a potentially important insight for designing communication efficient multi-agent protocols.

8. Limitations

The present model makes several simplifying assumptions that constrain its immediate applicability:

• **Mean field approximation.** All to all coupling suppresses network topology effects that arise in sparse or structured graphs.

• **Scalar memory.** The memory variable M_i is a scalar; richer tensor valued or associative memory models may be more realistic.

• **Linear memory dynamics.** The leaky integrator is a first order approximation; nonlinear saturation or adaptation are not modeled.

• **No adaptive learning.** β and λ are fixed parameters rather than dynamically adapted based on performance.

• **Simulations proposed, not executed.** The numerical experiments described in §5 constitute testable predictions; empirical confirmation is left to follow-up work.

9. Future Work

Natural extensions of this framework include: (i) graph structured coupling where K_{ij} reflects network adjacency; (ii) adaptive memory dynamics where β itself evolves in response to coherence; (iii) integration with neural network layers as trainable oscillator banks; (iv) reinforcement learning agents whose reward modulates memory gain; and (v) extension to the Ott Antonsen manifold [3] for exact low dimensional reduction of the memory augmented system.

References

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